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On Quasi-interpolation with Radial Basis Functions

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DEDICATED TO THE MEMORY OF I. J. SCHOENBERG

It has been known since 1987 that quasi-interpolation with radial functions on the integer grid can be exact for certain order polynomials. If, however, we require that the basis functions of the quasi-interpolants be *finite* linear combinations of translates of the radial functions, then this can be done only in spaces whose dimension has a prescribed parity. In this paper we show how infinite linear combinations of translates of a given radial function can be found that provide polynomial exactness in spaces whose dimensions do not have this prescribed parity. These infinite linear combinations are of a simple form. They are, in particular, easier to find than the cardinal functions of radial basis function interpolation, which provide polynomial exactness in all dimensions. The techniques that are used in this work also give rise to some remarks about *interpolation* with radial functions both on the integers and on the nonnegative integers.

1. INTRODUCTION

Radial basis function methods offer a promising approach for interpolating and approximating functions and data in several dimensions. Given a continuous function $\phi\colon \mathbb{R}_+ \mapsto \mathbb{R}$, we employ the n-variate, spherically symmetric function $\phi(\|\cdot\|)$, the norm being Euclidean, and denote it by Φ. Then, a radial basis function approximation is of the form

$$s(x) = \sum_j \lambda_j \Phi(x - x_j), \qquad x \in \mathbb{R}^n.$$

Here, $\{\lambda_j\}$ are real coefficients and $\{x_j\} \subset \mathbb{R}^n$ are distinct points ("centres") which usually coincide with the sampling points of the function to be approximated. Due to their well-known suitability for practical applications, the following radial basis functions are of special interest:

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$\phi(r)=r^{2k+1}$, $k=0, 1, 2, ...$, the multiquadric function $\phi(r)=\sqrt{r^2+c^2}$, c being a positive parameter, and its reciprocal, the inverse multiquadric function, and finally $\phi(r)=r^{2k}\log r$, $k=1, 2, ...$, the choice $k=1$ giving the so-called thin-plate-spline radial function. In order to evaluate the efficacy of approximations from linear spaces which are spanned by translates of radial functions, the case when the centres are all multi-integers, i.e., $\{x_j\}=\mathbb{Z}^n$, has been investigated extensively in the literature. Many of the methods that are applied in the analysis of approximations with centres on $\mathbb{Z}^n$ have been motivated by Schoenberg's techniques in his seminal work on univariate cardinal splines [17, 18]. Jackson [10, 11], for instance, studied *quasi-interpolants*

$$Q: f \mapsto \sum_{j\in\mathbb{Z}^n} f(j)\,\psi(\cdot - j), \tag{1.1}$$

f being the function we wish to approximate, and (1.1) is often called the *Schoenberg operator*. Here, ψ is usually a finite linear combination of integer translates of a radial function, and ψ is required to satisfy some minimal requirements which exclude choices that would make (1.1) useless for approximation. For example, an appropriate condition is that (1.1) be exact for constants, i.e., $Qf\equiv f$ when $f\equiv 1$. Formula (1.1) gives this property if ψ satisfies the conditions

$$\sum_{j\in\mathbb{Z}^n} |\psi(x-j)| < \infty \qquad \text{and} \qquad \sum_{j\in\mathbb{Z}^n} \psi(x-j)=1 \qquad \text{for all} \quad x\in\mathbb{R}^n. \tag{1.2}$$

Radial basis functions that provide such ψ's exist in abundance. For instance, Jackson [11] proved that the linear radial function $\phi(r)=r$ has the remarkable property that in *odd-dimensional* spaces, finite linear combinations ψ of translates of Φ can be found which render (1.1) well defined and exact when f is any polynomial in $\mathbb{P}_n^n$. (By $\mathbb{P}_n^m$ we denote the set of all polynomials in n variables of total degree at most m.) He then pursued a convergence analysis of quasi-interpolants

$$Q_h: f \mapsto \sum_{j\in\mathbb{Z}^n} f(jh)\,\psi(h^{-1}\cdot - j) \tag{1.3}$$

on an integer lattice scaled by h, and he found that $\phi(r)=r$ provides convergence orders $O(h^{n+1})$ to sufficiently differentiable functions f that have certain derivatives bounded. The same polynomial recovery is achieved if we use the multiquadric radial function instead, cf. [2]. Moreover, Powell [14] shows linear polynomial precision for multiquadrics in one dimension when the centres are no longer required to lie on a grid (but there still have to be infinitely many of them and he assumes that they tend to both $+\infty$ and $-\infty$). In the case when the centres are the multi-integers in $\mathbb{R}^n$, corresponding results are known for all the other radial functions we have mentioned; when $\phi(r)=r^{2k+1}$ we have polynomial recovery in odd-dimensional spaces for polynomials in $\mathbb{P}_n^{2k+n}$ and, when $\phi(r)=r^{2k}\log r$, we have exactness for polynomials of one order less in even-dimensional spaces [8]. Regarding the inverse multiquadric function, it is shown in [2] that exactness can be obtained for polynomials in $\mathbb{P}_n^{n-2}$ when $n=3, 5, \ldots$. It is known [10, 8], however, that there are no finite linear combinations of integer translates of the aforementioned radial functions that enjoy the property (1.2) if the dimension of the underlying space is of a parity other than the one specified above (or $n=1$ for inverse multiquadrics [2]). It is also known that *infinite* linear combinations that give the same polynomial recovery (i.e., $n+2k$ for $\phi(r)=r^{2k+1}$, n for multiquadrics, etc.) in even- and odd-dimensional spaces do exist for all the radial functions mentioned (but $n=1$ for inverse multiquadrics is still excluded). They are the *cardinal functions*

$$\chi(x)=\sum_{j\in\mathbb{Z}^n} d_j\Phi(x-j), \qquad x\in\mathbb{R}^n,$$

that satisfy $\chi(j)=\delta_{0j}$ for all multi-integers j and that therefore admit interpolation to functions on the integer lattice by

$$I: f\mapsto \sum_{j\in\mathbb{Z}^n} f(j)\,\chi(\cdot-j),$$

as described in [2] and [4] (also in [3] and [13], but there the previous restrictions to the parity of the spatial dimension still apply).

In the present paper we show that one need not work so hard as to obtain cardinal functions, in order to perform quasi-interpolation with polynomial recovery, in those spaces which do not include any finite linear combinations ψ that have the property (1.2). Specifically, for a class of radial functions containing all the ones mentioned above, we will construct functions ψ_0, which are finite linear combinations of translates of Φ, such that the function

$$\psi(x)=\sum_{j\in\mathbb{Z}^n} c_j\psi_0(x-j), \qquad x\in\mathbb{R}^n, \tag{1.4}$$

enjoys the polynomial recovery properties of the cardinal functions, where the coefficients $\{c_j\}_{j\in\mathbb{Z}^n}$ are of a simple form. They are, in particular, independent of which radial function is chosen from the class. This technique was motivated by the interesting article by Chui, Jetter, and Ward [7], who suggest a similar approach to construct ψ's that may not be good enough for quasi-interpolation and polynomial recovery (their basis functions are only shown to be in $L^2(\mathbb{R}^n)$), but one can generate cardinal functions for interpolation by applying a Neumann series method to linear combinations of translates of ψ.

In our work, we are mainly interested in the cases where the cardinal function does not decay exponentially (such as the multiquadric and the inverse multiquadric radial functions, or $\phi(r)=r^{2k+1}$ in even dimensions, cf. [2] and [4]), because in the event of exponential decay of the cardinal function (e.g., when $\phi(r)=r^{2k+1}$ and n is odd), interpolation always has a clear advantage over quasi-interpolation, where in almost all instances exponential decay cannot be achieved. On the other hand, if there is no exponential decay, the basis functions for quasi-interpolation can be more useful than cardinal functions because either they consist of only a finite number of terms or, as we mentioned in the previous paragraph, they are an infinite linear combination (with fixed coefficients which are independent of ϕ and only have to be computed once to high precision) of a function ψ_0 which again contains only finitely many terms. Rabut [16] has also studied quasi-interpolation in those spaces which do and in those spaces which do not contain finite linear combinations ψ that enjoy property (1.2). His work is restricted to "polyharmonic B-splines" (including $\phi(r)=r^{2k+1}$, $\phi(r)=r^{2k}\log r$, excluding multiquadrics, and inverse multiquadrics, for instance). We will come back to this at the end of the next section.

The construction of the aforementioned basis functions for quasi-interpolation will occupy the next section, where we address the described instances where finite linear combinations are not admissible to form ψ, except that the inverse multiquadric case in one dimension receives special treatment in the third section. This consideration of quasi-interpolation with inverse multiquadrics raises several questions about interpolation on the integer lattice in one dimension and on the nonnegative integers in one dimension that are also related to Schoenberg's work on cardinal splines [18] and that are studied in the third section.

2. Quasi-Interpolation in Even- and Odd-Dimensional Spaces

The following fact is well known (see, for instance, [2] or [8]), but we include it for completeness and because its proof contains elements which are crucial to the proof of our main result. From now on, $\hat{\Phi}$ denotes the generalized Fourier transform of the n-variate, radially symmetric function Φ, so that $\hat{\Phi}$ is also radially symmetric, and we denote its radial part by $\hat{\phi}$.

Theorem 1. *Let m be an even positive integer and let ϕ be a radial* *function of at most polynomial growth with* $\hat{\phi} \in \mathscr{C}^{m+n}(\mathbb{R}_{>0})$, $|\hat{\phi}^{(l)}(r)| = O(r^{-n-\varepsilon})$, $r > 1$, $\varepsilon > 0$, $l = 0, 1, \ldots, m+n$. *Suppose also that*

$$\hat{\phi}(r) = \sum_{j=0}^{m/2} a_j r^{2j-m} + b \log r + O(r), \qquad 0 < r < 1, \tag{2.1}$$

where $a_0 \neq 0$, *and that the lth derivative of the part of* $\hat{\phi}$ *that is absorbed in the last term on the right-hand side of* (2.1) *is bounded by a constant multiple of* r^{1-l} *on* $(0, 1)$ *for* $l = 1, 2, \ldots, m+n$. *Then there exist a finite subset* $N \subset \mathbb{Z}^n$ *and* $\{\mu_j\}_{j \in N}$ *such that*

$$\psi(x) = \sum_{j \in N} \mu_j \Phi(x - j), \qquad x \in \mathbb{R}^n, \tag{2.2}$$

satisfies

$$|\psi(x)| = O(\|x\|^{-n-m}),$$

and the quasi-interpolation formula (1.1) *is exact for all polynomials* $f \in \mathbb{P}_n^{m-1}$.

We note that this theorem applies to $\phi(r) = r^{2k+1}$ in odd-dimensional spaces, in which case $m = 2k + n + 1$, to the multiquadric radial function and its reciprocal when n is odd ($n > 1$ for inverse multiquadrics), where $m = n + 1$ and $m = n - 1$, respectively, and to the thin-plate-spline radial functions of order $2k$ in even dimensions, where $m = n + 2k$. This assertion is evident for $\phi(r) = r^{2k+1}$ and $\phi(r) = r^{2k} \log r$ since their distributional Fourier transforms are multiples of $r^{-n-2k-1}$ and r^{-n-2k}, respectively. In order to apply the theorem to the multiquadric radial function $\phi(r) = \sqrt{r^2 + c^2}$, we note that

$$\hat{\phi}(r) = -\pi^{-1}(2\pi c/r)^{(n+1)/2} K_{(n+1)/2}(cr), \qquad r > 0, \tag{2.3}$$

(see [9, p. 365]), $K_{(n+1)/2}$ being a modified Bessel function. This Bessel function is infinitely differentiable away from zero, it decays exponentially for large argument, and so do all its derivatives. It can be expanded in an ascending series [1, p. 375], the expansion having the form

$$\begin{aligned} K_{(n+1)/2}(z) = {} & \tfrac{1}{2}(\tfrac{1}{2}z)^{-(n+1)/2} \sum_{k=0}^{(n-1)/2} \frac{(\frac{1}{2}(n-1)-k)!}{k!} (-\tfrac{1}{4}z^2)^k \\ & + (-1)^{(n+3)/2} \frac{(\frac{1}{2}z)^{(n+1)/2}}{\Gamma(\frac{1}{2}(n+3))} \log \tfrac{1}{2}z \\ & + \tfrac{1}{2}(-1)^{(n+1)/2} \frac{\sum_{k=1}^{(n+1)/2} k^{-1} - 2\gamma}{\Gamma(\frac{1}{2}(n+3))} (\tfrac{1}{2}z)^{(n+1)/2} \\ & + O(z^{(n+3)/2}), \qquad z \to 0, \end{aligned} \tag{2.4}$$

where γ is Euler's constant. Therefore, ϕ satisfies the assumptions of the theorem. In the case of the inverse multiquadric function,

$$\hat{\phi}(r) = 2(2\pi c/r)^{(n-1)/2} K_{(n-1)/2}(cr), \qquad r > 0,$$

where $K_{(n-1)/2}$ is still a modified Bessel function. We see that the theorem does not apply when $n = 1$ since $K_0(z) \sim -\log z$ near 0.

The following result of Jackson [11] was instrumental to all his conclusions about the approximational power of radial functions and is of fundamental importance to our proof of Theorem 1.

LEMMA 1. *Let $\psi \in \mathscr{C}(\mathbb{R}^n)$ satisfy*

$$|\psi(x)| = O(\|x\|^{-n-k-\varepsilon}) \tag{2.5}$$

for some nonnegative integer k and some positive ε and let its Fourier transform

$$\hat{\psi}(t) = \int_{\mathbb{R}^n} \exp(-it \cdot x)\, \psi(x)\, dx, \qquad t \in \mathbb{R}^n,$$

satisfy

$$\left[\frac{\partial^{|\alpha|}\hat{\psi}(t)}{\partial t_1^{\alpha_1} \partial t_2^{\alpha_2} \cdots \partial t_n^{\alpha_n}}\right]_{t = 2\pi l} = \delta_{0l}\,\delta_{0\alpha} \qquad \text{for all} \quad 0 \leqslant |\alpha| \leqslant k \quad \text{and} \quad l \in \mathbb{Z}^n. \tag{2.6}$$

Then (1.1) *is exact for all $f \in \mathbb{P}_n^k$.*

In Jackson's paper, Lemma 1 is proved only for $\varepsilon = 1$ (this is his Theorem 3), but it is straightforward to show that it holds for all positive ε. This result was motivated by a similar, though univariate, result due to Schoenberg [17]. Now, briefly, the proof of the first theorem.

Proof of Theorem 1. Let us denote the interval $[-\pi, \pi]$ by $\mathbb{T}$. Let g be the trigonometric polynomial

$$g(t) = \sum_{j \in N} \mu_j \exp(-it \cdot j), \qquad t \in \mathbb{T}^n,$$

for some finite set $N \subset \mathbb{Z}^n$ and coefficients $\{\mu_j\}_{j \in N} \subset \mathbb{R}$ both of which are yet to be chosen. We may expand g near the origin as

$$g(t) = \sum_{s=0}^{\infty} \sum_{j \in N} \mu_j (-it \cdot j)^s / s! = \sum_{s=0}^{\infty} P_s(t),$$

where each P_s is a homogeneous polynomial of degree s in n variables. We shall define ψ via its Fourier transform and the product $g\hat{\Phi}$ is going to be that Fourier transform. Therefore, in order to render ψ real-valued, we require $P_s = 0$ for all odd s, a requirement that is fulfilled if $N = -N$ and $\mu_j = \mu_{-j}$ for all $j \in N$. We shall now seek additional conditions on N and $\{\mu_j\}_{j\in N}$ so that the function $\tilde{\psi} := g\hat{\Phi}$ is in $\mathscr{C}^{m-1}(\mathbb{R}^n)$ and satisfies (2.6) for $\tilde{\psi}$ replacing $\hat{\psi}$ and $k = m-1$. Thereafter we let ψ be the inverse Fourier transform of $\tilde{\psi}$ and we show that ψ satisfies (2.5) for $k = m-1$ and $\varepsilon = 1$. Then we deduce from the absolute integrability of ψ that $\hat{\psi} = \tilde{\psi}$. Thus ψ enjoys all the requirements of Lemma 1 and has the form (2.2), the exponential term in the definition of g providing the shifts of formula (2.2).

To begin with, let us assume without loss of generality that $a_0 = 1$. We require that N and $\{\mu_j\}_{j\in N}$ be such that $P_s = 0$ for all $s < m$ and $P_m = \|\cdot\|^m$, and we note that thus $\tilde{\psi}$ is continuous and absolutely integrable and $\tilde{\psi}(t) = 1 + o(1)$ near the origin. Further, denoting $\|t\|$ by r, we have near zero

$$\begin{aligned}
\tilde{\psi}(t) &= \sum_{s=m/2}^{\infty} P_{2s}(t)\left\{\sum_{j=0}^{m/2} a_j r^{2j-m} + b\log r + O(r)\right\}\\
&= \sum_{s=m/2}^{m} P_{2s}(t)\left\{\sum_{j=0}^{m/2} a_j r^{2j-m} + b\log r\right\} + O(r^{m+1})\\
&= 1 + \{r^{-m}P_{m+2}(t) + a_1 r^2\}\\
&\quad + \{r^{-m}P_{m+4}(t) + a_1 r^{2-m}P_{m+2}(t) + a_2 r^4\} + \cdots\\
&\quad + \{r^{-m}P_{2m-2}(t) + \cdots + a_{m/2-1}r^{m-2}\}\\
&\quad + \{r^{-m}P_{2m}(t) + \cdots + a_{m/2}r^m\} + br^m\log r + O(r^{m+1}).
\end{aligned}$$

Because of the assumptions on $\hat{\phi}$, and because g is a 2π-periodic function with an m-th order zero at the origin, $\tilde{\psi}$ has the properties (2.6) for $k = m-1$ whenever all the terms in braces except the last one in the penultimate line of the above displayed equation are eliminated. We achieve this condition by defining $P_{m+2}, P_{m+4}, \ldots, P_{2m-2}$ recursively in a suitable way, i.e.,

$$\begin{aligned}
P_{m+2}(t) &= -a_1 r^{m+2},\\
P_{m+4}(t) &= -a_1 r^2 P_{m+2}(t) - a_2 r^{m+4},
\end{aligned}$$

etc. Thus each $P_s(t)$ is a multiple of r^s and we also let $P_{2m}(t)$ be a multiple of r^{2m}. This construction of each P_s provides our conditions on the $\{\mu_j\}_{j\in N}$, which can be satisfied for large enough N by the linear independence of polynomials on $\mathbb{Z}^n$.

We now have to demonstrate that the inverse Fourier transform

$$\psi(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \exp(x\cdot t)\,\tilde{\psi}(t)\,dt, \qquad x \in \mathbb{R}^n,$$

satisfies (2.5) for $k=m-1$. By our requirements on g, we have, still replacing $\|t\|$ by r,

$$\tilde{\psi}(t)=1+\{r^{-m}P_{2m}(t)+\cdots+a_{m/2}r^{m}\}+br^{m}\log r+O(r^{m+1}),\qquad 0<r<1. \tag{2.7}$$

Recall that $\tilde{\psi}\in\mathscr{C}^{m+n}(\mathbb{R}^n\setminus\{0\})$ and that $\tilde{\psi}$ and its partial derivatives of total order at most $m+n$ are integrable about $\mathbb{R}^n\setminus B(0)$ and vanish at infinity, where $B(0)$ can be any ball about zero of positive radius. Hence, we alter the asymptotic behaviour of the inverse Fourier transform of $\tilde{\psi}$ only by a term which is $o(\|x\|^{-m-n})$ for large $\|x\|$ when we replace $\tilde{\psi}$ by $\rho(\|\cdot\|^2)\,\tilde{\psi}$, ρ being nonnegative, in $\mathscr{C}^{\infty}(\mathbb{R})$, with support in $(-1,1)$, and $\rho|_{[-1/2,1/2]}\equiv 1$. Therefore, we only have to study the contributions to $|\psi(x)|$ for large $\|x\|$ that come from the terms which appear in (2.7). The first two terms therein are infinitely often continuously differentiable and therefore contribute a rapidly decaying term to $|\psi(x)|$ for large $\|x\|$. The $O(r^{m+1})$ term in (2.7) gives a contribution of $o(\|x\|^{-m-n})$ to $|\psi(x)|$ for large $\|x\|$, because the part of $\tilde{\psi}$ it corresponds to is an absolutely integrable function that vanishes at infinity whose derivatives of order up to $m+n$ are absolutely integrable and vanish at infinity too. Finally, the distributional Fourier transform of $r^m\log r$ is of magnitude $\|x\|^{-n-m}$ at infinity [9, p. 364]. By standard techniques applied in [4], it follows that this term gives a contribution of magnitude $\|x\|^{-m-n}$ to the asymptotic behaviour of $|\psi(x)|$ at infinity. Thus we conclude that ψ satisfies all the conditions of Lemma 1 for $k=m-1$, which completes the proof. ∎

We remark that it follows from this proof [specifically, from expression (2.7)] that the dominant term in the asymptotic estimate (2.5) is

$$(-1)^{m/2-1}\,b\pi^{-n/2}2^{m-1}\Gamma(\tfrac{1}{2}n+\tfrac{1}{2}m)\,\Gamma(\tfrac{1}{2}m+1)\|x\|^{-n-m}$$

if $b\neq 0$, where b is a constant from (2.1) and where we have taken the Fourier transform of $\|x\|^m\log\|x\|$ from Gel'fand and Shilov [9]. If, on the other hand, $b=0$, then the rate of decay of ψ that can be established depends on the smoothness of the term that is incorporated in $O(r)$ in (2.1), but it is always at least $o(\|x\|^{-m-n})$. For instance, if there are no terms in $\hat{\phi}$ other than the finite sum over $a_j r^{2j-m}$, then any fixed algebraic rate of decay can be established by choosing the μ_j judiciously. The point here is that $\tilde{\psi}(t)=1+O(r^{\tilde{m}+1})$, $0<r<1$, can be achieved, for any even $\tilde{m}$, instead of (2.7). This is done by letting $P_s=0$, $s<m$, by considering the expansion

$$\tilde{\psi}(t)=\sum_{s=m/2}^{(m+\tilde{m})/2}P_{2s}(t)\left\{\sum_{j=0}^{m/2}a_j r^{2j-m}\right\}+O(r^{\tilde{m}+1})$$

and by choosing the polynomials $P_m, P_{m+2}, ..., P_{m+\tilde{m}}$ according to the same principles as demonstrated above, i.e., to make them eliminate all terms (except 1) of order less than $\tilde{m}+1$. Thus $\tilde{\psi}$ is endowed with such high smoothness that $|\psi(x)| = O(\|x\|^{-n-\tilde{m}})$ can be established in the usual way. We record this observation in

Remark 1. If the assumptions of the previous theorem hold and if $\hat{\phi}$ only consists of the terms $a_j r^{2j-m}$ that appear in (2.1), then by judicious choice of the μ_j's, ψ can be endowed with any integral rate of decay at infinity.

It follows from the work in [2–4] that for all the radial functions that have been mentioned after the statement of Theorem 1 to which the theorem applies, the cardinal functions for interpolation decay faster than the rate that has been established for the functions (2.2). For example, for $\phi(r) = r^{2k+1}$ and odd n, the cardinal functions decay exponentially whereas only algebraic decay of ψ has been found here (this, however, may be arbitrarily high, as we have just pointed out). Moreover, for the multiquadric function or the inverse multiquadric function, the cardinal functions $\chi(x)$ decay as $\|x\|^{-3n-2}$ and $\|x\|^{-3n+1}$, respectively, but Theorem 1 only gives decay as $\|x\|^{-2n-1}$ and $\|x\|^{-2n+1}$, respectively. Of course the statement of Theorem 1 does not preclude the possibility that ψ's of the form (2.2) may be found which decay faster than the cardinal functions, but it has been shown that this is not so [8, 10].

We now state our principal result.

THEOREM 2. *Let m be an odd positive integer and let ϕ be a radial function of at most polynomial growth with $\hat{\phi} \in \mathscr{C}^{m+n}(\mathbb{R}_{>0})$, $|\hat{\phi}^{(l)}(r)| = O(r^{-n-\varepsilon})$, $r > 1$, $\varepsilon > 0$, $l = 0, 1, ..., m+n$. Suppose also*

$$\hat{\phi}(r) = \sum_{j=0}^{(m-1)/2} a_j r^{2j-m} + b + c \log r + O(r), \qquad 0 < r < 1, \tag{2.8}$$

where $a_0 \neq 0$, and that the lth derivative of the part of $\hat{\phi}$ that is absorbed in the last term on the right-hand side of (2.8) *is bounded by a constant multiple of r^{1-l} on $(0, 1)$ for $l = 1, 2, ..., m+n$. Then there exists a finite subset $N \subset \mathbb{Z}^n$ and $\{\mu_j\}_{j\in N}$ such that*

$$\psi_0(x) = \sum_{j\in N} \mu_j \Phi(x-j), \qquad x \in \mathbb{R}^n, \tag{2.9}$$

satisfies

$$|\psi_0(x)| = \begin{cases} O(\log |x|) & \text{if } n = 1, \\ O(\|x\|^{1-n}) & \text{if } n > 1, \end{cases} \qquad \|x\| > 2. \tag{2.10}$$

Further defining $\{c_j\}_{j\in\mathbb{Z}^n}$ *to be the Fourier coefficients of the function*

$$\tilde{g}(t)=\left\{\sum_{k=1}^{n}\left[2-2\cos t_k\right]\right\}^{1/2},\qquad t\in\mathbb{T}^n, \tag{2.11}$$

where $\{t_k\}_{k=1}^n$ *are the components of* t, *the sum* (1.4) *is absolutely convergent and satisfies the requirements of Lemma* 1 *for* $k=m-1$. *Therefore* (1.1) *recovers all polynomials of total order less than* m.

This theorem applies to all the radial functions that have been mentioned already if the dimension of the underlying space is complementary to the dimension that occurs in Theorem 1 (with the exception of inverse multiquadrics in one dimension). This assertion is obvious in the case of odd powers and radial functions of the thin-plate-spline type. In the case of multiquadrics and inverse multiquadrics, however, we draw this conclusion from equations (2.3) and (2.4) and the expression

$$\begin{aligned}\frac{K_{(n+1)/2}(z)}{\sqrt{\frac{1}{2}\pi}}=\frac{1\cdot 3\cdot 5\cdots(n-1)}{z^{(n+1)/2}}\Bigg\{1&+\frac{\frac{1}{2}z^2}{1!(1-n)}\\ &+\frac{(\frac{1}{2}z^2)^2}{2!(1-n)(3-n)}+\frac{(\frac{1}{2}z^2)^3}{3!(1-n)(3-n)(5-n)}\\ &+\cdots+\frac{(\frac{1}{2}z^2)^{n/2}}{(\frac{1}{2}n)!\,(1-n)(3-n)\cdots(-1)}\Bigg\}\\ &+\frac{(-z)^{(n+1)/2}}{1\cdot 3\cdot 5\cdots(n+1)}+O(z^{(n+3)/2}),\qquad z\to 0,\end{aligned}$$

which is valid when n is even [1, p. 443, formulae 10.2.4–10.2.6].

It is worth noticing that the coefficients $\{c_j\}_{j\in\mathbb{Z}^n}$ in Theorem 2 are independent of our choice of ϕ. (We have mentioned this in the introduction, but it is so important that it certainly bears repeating.) It is in this respect that the basis functions ψ are simpler than the cardinal functions for interpolation, because the $\{c_j\}_{j\in\mathbb{Z}^n}$ can be identified (and tabulated to high precision) once and forever, and so for the individual choices of ϕ we only have to find the finite number of coefficients $\{\mu_j\}_{j\in N}$. This is in contrast to the cardinal case where for each new ϕ a full set of infinitely many coefficients must be found.

Proof of Theorem 2. The proof consists of two basic steps. First, we identify a trigonometric polynomial g such that $\tilde{\psi}$, which is now defined as the product $\tilde{\psi}=g\tilde{g}\hat{\Phi}$, has an inverse Fourier transform for which the conditions of Lemma 1 hold when $k=m-1$. Second, we show that the inverse Fourier transform ψ_0 of $g\hat{\Phi}$ has the asymptotic properties (2.10). This, together with the observation that $|c_j| = O(\|j\|^{-n-1})$, which is deduced from the definition of $\tilde{g}$, implies the absolutely convergence of the sum in (1.4). In other words, (2.10) allows for the relatively slow decay of the $\{c_j\}_{j\in\mathbb{Z}^n}$ to be in unison with absolute convergence of the series (1.4).

Since the estimate $|c_j| = O(\|j\|^{-n-1})$ also implies that $\tilde{g}$ equals its Fourier expansion, i.e.,

$$\tilde{g}(t) = \sum_{j\in\mathbb{Z}^n} c_j \cos(t\cdot j), \qquad t\in\mathbb{T}^n$$

[19, Corollary 1.8, p. 249], it follows that the function (1.4), namely ψ, is the inverse Fourier transform of $\tilde{\psi}$, so ψ has the required properties.

It transpires that the purpose of the trigonometric polynomial g is to resolve the even integral part of $\hat{\phi}$'s singularity at the origin whereas $\tilde{g}$ resolves the remaining singularity. It should be pointed out that other 2π-periodic functions which have a zero of single order at the origin and are smooth enough would do (instead of $\tilde{g}$) as well.

The main objective of the first step of our proof is to achieve the relation $g(t)\,\tilde{g}(t)\,\hat{\Phi}(t) = 1 + o(\|t\|^{m-1})$ when $\|t\|$ is small. Let us assume again that a_0 is 1. Since we have $\tilde{g}(t) = \|t\| + O(\|t\|^3)$, we can rewrite the required equation as

$$g(t)\,\hat{\Phi}(t) = [\tilde{g}(t)]^{-1} + o(\|t\|^{m-2}). \tag{2.12}$$

A Taylor expansion of the left-hand side of Eq. (2.12) is

$$\left\{\sum_{s=m-1}^{2m-2} P_s(t)\right\}\{\|t\|^{-m} + a_1\|t\|^{2-m} + \cdots + a_{(m-1)/2}\|t\|^{-1}\} + o(\|t\|^{m-2})$$

if we require that

$$P_s \equiv 0 \qquad \text{for all} \quad s = 0, 1, ..., m-2, \tag{2.13}$$

(this is a vacuous condition if $m = 1$) while we can write the right-hand side of (2.12) as

$$\begin{aligned}
&[\tilde{g}(t)]^{-1} + o(\|t\|^{m-2})\\
&\quad = \left\{\sum_{k=1}^{n}[2-2\cos t_k]\right\}^{-1/2} + o(\|t\|^{m-2})\\
&\quad = \|t\|^{-1}\left\{1 + \frac{2}{\|t\|^2}\left[-\frac{1}{4!}\sum_{k=1}^{n} t_k^4 + \frac{1}{6!}\sum_{k=1}^{n} t_k^6\right.\right.\\
&\qquad \left.\left. -\frac{1}{8!}\sum_{k=1}^{n} t_k^8 + \cdots\right]\right\}^{-1/2} + o(\|t\|^{m-2})\\
&\quad = \|t\|^{-1}[1 + Q_2(t) + Q_4(t) + \cdots + Q_{m-1}(t)] + o(\|t\|^{m-2}),
\end{aligned}$$

where every $\|t\|^{m-1} Q_j(t)$ is a homogeneous polynomial of degree $m-1+j$. A comparison of the above expansions shows that we need

$$P_{m-1}(t) = \|t\|^{m-1}. \tag{2.14}$$

Moreover, we also demand that

$$P_j(t) \equiv 0 \qquad \text{for all odd integers } j. \tag{2.15}$$

We see further that P_{m+1} must satisfy

$$\|t\|^{-m} P_{m+1}(t) + a_1 \|t\|^{2-m} P_{m-1}(t) = \|t\|^{-1} Q_2(t)$$

so $P_{m+1}(t) = \|t\|^{m-1} Q_2(t) - a_1 \|t\|^2 P_{m-1}(t)$, which is a homogeneous polynomial of degree $m+1$. Further, when l is in the range $m+1$, $m+3, ..., 2m-2$, we deduce the general formula

$$P_l(t) = \|t\|^{m-1} Q_{l-m+1}(t) - \sum_{j=m-1, m+1, ..., l-2} a_{(l-j)/2} \|t\|^{l-j} P_j(t), \tag{2.16}$$

which is again vacuous if $m = 1$.

If the P_s satisfy relations (2.13)–(2.16), then $\tilde{\psi}$ satisfies (2.6) when $m-1$ replaces k and $\tilde{\psi}$ replaces $\hat{\psi}$. That the inverse Fourier transform ψ of $\tilde{\psi}$ satisfies (2.5) for $k = m-1$ can be established in the same way as in the previous proof, noting that (2.13)–(2.16) imply

$$\tilde{\psi}(t) = 1 + b\|t\|^m + c\|t\|^m \log \|t\| + O(\|t\|^{m+1})$$

for small $\|t\|$, using the fact that the distributional Fourier transforms of $\|t\|^m$ and $\|t\|^m \log \|t\|$ are of magnitude $\|x\|^{-n-m}$ and $\|x\|^{-n-m} \log \|x\|$, respectively, and observing that the contributions from the loss of smoothness of $\tilde{\psi}$ at the non-zero multi-integer multiples of 2π are no worse than the contribution that comes from $\tilde{\psi}$'s behaviour in a neighbourhood of zero. We can give the precise form of the dominant term in the asymptotic expansion of ψ by taking the Fourier transforms from Gel'fand and Shilov [9] explicitly. It is

$$\begin{aligned} &\pi^{-n/2} 2^m [\Gamma(\tfrac{1}{2}n + \tfrac{1}{2}m)/\Gamma(-\tfrac{1}{2}m)] \Bigg[\Bigg(\tfrac{1}{2} c\xi(\tfrac{1}{2}n + \tfrac{1}{2}m - 1) + \tfrac{1}{2} c\xi(-\tfrac{1}{2}m - 1) \\ &\qquad + b + \sum_{l \in \mathbb{Z}^n \setminus \{0\}} \cos(2\pi l \cdot t)\, \hat{\Phi}(2\pi l) \Bigg) \\ &\qquad \times \|x\|^{-m-n} + c\|x\|^{-m-n} \log \|x\| \Bigg], \end{aligned} \tag{2.17}$$

where ξ denotes the Digamma-function (see [1]). It is usually denoted by ψ, but we have opted for this non-standard notation here for obvious reasons.

We now address the asymptotic behaviour of ψ_0 for large argument, and so we study the behaviour of $g\hat{\Phi}$ near the origin. The following expansion is useful for that purpose:

$$\begin{aligned} g(t)\,\hat{\Phi}(t) &= \left\{ \|t\|^{m-1} + \sum_{s=(m+1)/2}^{m-1} P_{2s}(t) \right\} \\ &\quad \times \left\{ \|t\|^{-m} + \sum_{j=1}^{(m-1)/2} a_j \|t\|^{2j-m} + b + c \log \|t\| \right\} + O(\|t\|^m) \\ &= \|t\|^{-1} + O(-\log \|t\|), \qquad 0 < \|t\| < 1. \end{aligned} \tag{2.18}$$

The expansion (2.18), together with the periodicity of g and the decay of $\hat{\phi}$ at infinity, shows that $g\hat{\Phi}$ is integrable when $n > 1$. We can therefore define ψ_0 as the inverse Fourier transform thereof. Further, by similar arguments as have been applied in the last paragraph of the previous proof, $|\psi_0(x)| = O(\|x\|^{1-n})$ whenever $n > 1$. Here it is crucial to note that the distributional Fourier transform of $\|t\|^{-1}$ is a constant multiple of $\|x\|^{1-n}$ and that this is the term of (2.18) that gives the dominant contribution to $|\psi_0(x)|$. The explicit Fourier transform that contributes the dominant term is

$$\tfrac{1}{2}\pi^{-(n+1)/2}\Gamma(\tfrac{1}{2}(n-1))\,\|x\|^{1-n}.$$

In the univariate case, $g\hat{\Phi}$ is not integrable over the reals. However, the difference $g\hat{\Phi} - \bar{h}\,|\cdot|^{-1}$ is integrable, where $\bar{h}$ is the characteristic function of the interval $(-1, 1)$. Therefore, ψ_0 is the sum of the inverse Fourier transform of this integrable function plus the distributional inverse Fourier transform of $\bar{h}|\cdot|^{-1}$. The former is bounded and the latter is for large x of the same magnitude as the distributional inverse Fourier transform

$$-\frac{\gamma + \log|x|}{\pi}$$

of $|t|^{-1}$, given in [9, p. 361], with γ being Euler's constant. Hence $|\psi_0(x)| = O(\log|x|)$ for $|x| > 2$ when $n = 1$.

The indicated decay rate of the Fourier coefficients of (2.11) is established by employing an expansion of $\tilde{g}$ near the origin and by using the fact that the distributional Fourier transform of $\|t\|$ is a constant multiple of $\|x\|^{-n-1}$. Since the technique which needs to be applied here is similar to our analysis of the asymptotic behaviour of $g\hat{\Phi}$, we omit the details. In summary, we obtain that (1.4) is well-defined and supplies an integrable function, whose Fourier transform satisfies (2.6) for $k=m-1$, and which gives itself (2.5) (for $\varepsilon=\frac{1}{2}$, say). The theorem is proved. ∎

It is easy to see the previous theorem can be generalized to admit $m\in\mathbb{R}_+\setminus 2\mathbb{Z}_+$, where the decay estimate (2.10) has to be modified accordingly, viz.,

$$|\psi_0(x)|=\begin{cases}O(\log|x|) & \text{if } m \text{ is an odd integer and } n=1,\\ O(\|x\|^{-n+m-2M}) & \text{in all other cases,}\end{cases}\qquad \|x\|>2,$$

where M is the largest integer less than $\frac{1}{2}m$, and where we now define

$$\tilde{g}(t)=\left\{\sum_{k=1}^{n}[2-2\cos t_k]\right\}^{m/2-M},\qquad t\in\mathbb{T}^n,$$

and ψ satisfies the requirements of Lemma 1 with the integer k in the statement of Lemma 1 being the largest integer less than m.

We want to include a remark about the distance between our quasi-interpolating basis functions and the cardinal functions. This distance can be conveniently measured by observing that we can bound

$$\begin{aligned}\|\chi-\psi\|_\infty &\leqslant\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\hat{\Phi}(t)\,|\sigma(t)^{-1}-g^*(t)|\,dt\\ &=\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}\sigma(t)\,|\sigma(t)^{-1}-g^*(t)|\,dt,\end{aligned}$$

where we assume that $\hat{\psi}=g^*\hat{\Phi}$ and $\hat{\Phi}\geqslant 0$, and where

$$\sigma(t)=\sum_{l\in\mathbb{Z}^n}\hat{\Phi}(t+2\pi l),\qquad t\in\mathbb{T}^n.$$

Hence the distance can be viewed as the $L^1_\sigma(\mathbb{T}^n)$ distance between g^* and σ^{-1} where $L^1_\sigma(\mathbb{T}^n)$ is the space of integrable functions on the torus $\mathbb{T}^n$ with respect to the $L^1(\mathbb{T}^n)$-norm weighted with the nonnegative 2π-periodic function σ (see also [5]). If g^* is a trigonometric polynomial as in the context of Theorem 1, it is interesting to look at best approximations to σ^{-1} with respect to this norm or perhaps the $L^2_\sigma(\mathbb{T}^n)$-norm by trigonometric polynomials g^* whose coefficient sequences are supported on a finite subset of $\mathbb{Z}^n$, call it Ξ, and which are such that $|g^*(t)|=O(\|t\|^m)$ near the origin, i.e., whose coefficients satisfy certain moment conditions. It turns out that best $L^2_\sigma(\mathbb{T}^n)$-approximations are obtained if ψ can be modified by a polynomial of total degree less than m such that it satisfies the cardinality conditions on Ξ. If, on the other hand, $g^*=\tilde{g}g$, where $\tilde{g}$ is as in Theorem 2, one can look at the best approximation to σ^{-1} by $\tilde{g}g$, where g is a trigonometric polynomial whose coefficient sequence is supported on Ξ and which satisfies $|g(t)| = O(\|t\|^{m-1})$. If instead we want to measure the distance between ψ and χ by comparing their coefficients and if we are in the context of Theorem 1, we are looking at

$$\sum_{j \in \mathbb{Z}^n} |d_j - \mu_j|^2 = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} |g^*(t) - \sigma(t)^{-1}|^2 \, dt,$$

where the d_j are the coefficients of the cardinal function. This distance may be made small by making use of the decay of the d_j (see [2] and [4]) and identifying μ_j with d_j on a finite set of integers j where d_j is large, and then modifying the μ_j's so that they obey moment conditions according to the principles of the proof of Theorem 1. We shall not, however, discuss this any further because it is not our main concern here.

The work of Rabut [16] addresses the cases when the expansions (2.1) and (2.8) only contain the dominant term (i.e., r^{-m}), both for integral and non-integral m. Such ϕ's are called polyharmonic B-splines. He gives explicit expressions for the coefficients of ψ (in the form of Fourier coefficients of certain simple 2π-periodic functions). In the one-dimensional setting, he also evaluates those coefficients. He is able to do this because polyharmonic B-splines are linear combinations of shifts of fundamental solutions of the iterated Laplace operator, and this allows to view the coefficients μ_j as the coefficients of a discretization of the iterated Laplace operator. The Laplace operator that occurs can be taken to a power which need not be integral—if it is not, then the according operator is defined via its Fourier transform—and for that reason non-integral m is admissible. Coefficients of discrete iterated Laplace operators are suitable special cases of the μ_j we use in our work whenever we are dealing with polyharmonic B-splines, but the more detailed analysis we present here is necessary if other ϕ's are considered. Rabut obtains polynomial recovery for the polyharmonic B-splines of the same order as we do when m is even and integral, but only considers reproduction of linear polynomials otherwise.

At this point it is appropriate to give a theorem that links the polynomial recovery of quasi-interpolation on the integer lattice with convergence estimates of quasi-interpolants on a scaled grid to differentiable functions. We shall be satisfied with quoting the following theorem from [2], but we point out that sometimes stronger results can be proved that dispose of the $\log h$ term which occurs in the estimate (2.19) (see, for instance, [11]).

THEOREM 3. *Let ψ satisfy the conditions of Lemma 1 for a nonnegative integer k and let $f \in \mathscr{C}^{k+1}(\mathbb{R}^n)$ have bounded kth and $(k+1)$th total order partial derivatives. Then* (1.3) *supplies*

$$\|Q_h f - f\|_\infty = O(-h^{k+1} \log h), \qquad 0 < h < \tfrac{1}{2}. \tag{2.19}$$

3. A Look at the Inverse Multiquadric Radial Function in One Dimension

It has been shown in [2] that quasi-interpolation (1.1) in one dimension with ψ being a finite linear combination of translates of $\phi(r) = 1/\sqrt{r^2+c^2}$ is not possible. Indeed, the form of $\hat{\phi}$ given above and the expansion

$$K_0(z) = -\log \tfrac{1}{2}z - \gamma + O(z), \qquad z \to 0,$$

[1, p. 375], where γ is Euler's constant, show that one cannot multiply $\hat{\Phi}$ with an even trigonometric polynomial g and achieve that $(g\hat{\Phi})(0)$ is bounded and nonzero at the same time. Hence, by Theorem 6 in [11], which pertains to the necessity of the conditions (2.6) for (1.1) to recover polynomials of degree k, we cannot find a finite linear combination of translates of Φ which enjoys the property (1.2). Hence, in particular, Theorem 1 is not applicable to the present setting. However, the following assertion is true:

Theorem 4. *Let ϕ be a radial function with $|\phi(r)| = O(r^{-1})$, $\hat{\phi} \in \mathscr{C}^2(\mathbb{R}_{>0})$, $|\hat{\phi}^{(l)}(r)| = O(r^{-1-\varepsilon})$, $r > 1$, $\varepsilon > 0$, $l = 0, 1, 2$, and let it be such that*

$$\hat{\phi}(r) = -\log r + a + O(r), \qquad 0 < r < 1, \tag{3.1}$$

where the part of $\hat{\phi}$ that is contained in the last term on the right-hand side of (3.1) *remains bounded on* $(0, 1)$ *by a constant or by a fixed multiple of r^{-1} if it is differentiated once or twice, respectively. Then the Fourier coefficients $\{d_j\}_{j=-\infty}^{\infty}$ of*

$$g(t) = -2\{\log(2 - 2\cos t)\}^{-1} \sum_{l=-\infty}^{\infty} \rho((t + 2\pi l)^2), \qquad t \in \mathbb{T}, \tag{3.2}$$

where ρ is defined in the proof of Theorem 1, *are such that*

$$\psi(x) = \sum_{j=-\infty}^{\infty} d_j \phi(|x - j|), \qquad x \in \mathbb{R}, \tag{3.3}$$

is an absolutely convergent sum and supplies (1.2) *for $n = 1$.*

We point out that the $\{d_j\}_{j=-\infty}^{\infty}$ are not unique. Other 2π-periodic functions g with a logarithmic zero at the origin will do as well; in particular, other cut-off functions than the ρ defined in the proof of Theorem 1 could be used.

Proof. We note that g is continuous on $\mathbb{T}$ and hence it is in the set $L^2(\mathbb{T})$, by which we shall denote the set of square-integrable functions on

$\mathbb{T}$. It is therefore immediate that the Fourier coefficients of (3.2) render the right-hand side of (3.3) absolutely convergent, because they are square-summable (see [19, Theorem 1.7, p. 248]) and because an application of the Cauchy–Schwarz inequality gives

$$\sum_{j=-\infty}^{\infty} |d_j \phi(|x-j|)| \leqslant \left\{ \sum_{j=-\infty}^{\infty} |d_j|^2 \right\}^{1/2} \left\{ \sum_{j=-\infty}^{\infty} |\phi(|x-j|)|^2 \right\}^{1/2},$$

which is uniformly bounded.

Now, observe that for small t

$$\begin{aligned} g(t) &= -\left\{ \frac{1}{2} \log \left(t^2 - 2 \left(\frac{t^4}{4!} - \frac{t^6}{6!} + \cdots \right) \right) \right\}^{-1} \\ &= -\left\{ \log |t| + \frac{1}{2} \log \left(1 - 2 \left(\frac{t^2}{4!} - \frac{t^4}{6!} + \cdots \right) \right) \right\}^{-1} \\ &= -\frac{1}{\log |t|} \left\{ 1 + (2 \log |t|)^{-1} \log \left(1 - 2 \left(\frac{t^2}{4!} - \frac{t^4}{6!} + \cdots \right) \right) \right\}^{-1} \qquad (3.4) \\ &= -\frac{1}{\log |t|} + O\left(\frac{t^2}{(\log |t|)^2} \right), \qquad (3.5) \end{aligned}$$

where (3.5) was obtained by expanding the third log term in (3.4) and by expanding the term in braces in (3.4). Therefore, for small t,

$$g(t)\,\hat{\phi}(|t|) = 1 - \frac{a}{\log |t|} + O\left(-\frac{|t|}{\log |t|} \right). \qquad (3.6)$$

Moreover, $g(2\pi l) = 0$ for all integers l. Hence

$$\tilde{\psi} = g\hat{\Phi}$$

satisfies (2.6) for $k = 0$. Further, its inverse Fourier transform

$$\psi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ixt)\, \tilde{\psi}(t)\, dt, \qquad x \in \mathbb{R}, \qquad (3.7)$$

is well-defined and continuous. It has the form (3.3) because $g(t)$ is the same as its Fourier expansion $\sum_{j=-\infty}^{\infty} d_j \cos(jt)$ for all t by Theorem 15.3(ii) on page 58 of [12], which we can apply since the absolute integrability of g and of its first derivative imply that the Fourier coefficients $\{d_j\}_{j=-\infty}^{\infty}$ of g decay as $|j|^{-1}$. We have to show that ψ is absolutely integrable so that we may conclude $\hat{\psi} = \tilde{\psi}$. We claim that

$$|\psi(x)| = O\left(\frac{1}{|x|\,(\log |x|)^2} \right), \qquad |x| > 2. \qquad (3.8)$$

Estimate (3.8) shows that ψ is absolutely integrable, because

$$\int_2^\infty \frac{dx}{x(\log x)^2} = -\frac{1}{\log x}\bigg|_2^\infty = \frac{1}{\log 2}.$$

At the same time, this implies

$$\sum_{j=-\infty}^{\infty} |\psi(x-j)| < \infty \qquad \text{for all} \quad x \in \mathbb{R}, \tag{3.9}$$

because ψ is continuous and because

$$\sum_{j=3,4,5,\ldots} \frac{1}{j(\log j)^2} < \int_2^\infty \frac{dx}{x\,(\log x)^2} = \frac{1}{\log 2}.$$

Estimate (3.8) is not, however, sufficient for (2.5) to hold for $k=0$ and $n=1$, but Jackson's proof of Lemma 1 can be carried through for $n=1$ and $k=0$ when (3.8) holds instead of (2.5), because (3.9) is the salient property of ψ that is needed for the proof.

We note that integration by parts, applied to (3.7), gives, due to the symmetry of $\tilde{\psi}$,

$$|2\pi \cdot x \cdot \psi(x)| = \left|\int_{-\infty}^{\infty} \sin(xt)\,\tilde{\psi}'(t)\,dt\right|.$$

So, in order to establish (3.8), it is sufficient to show that

$$\left|\int_{-\infty}^{\infty} \sin(xt)\,\tilde{\psi}'(t)\,dt\right| = O\left(\frac{1}{(\log|x|)^2}\right), \qquad |x|>2. \tag{3.10}$$

The part of $g\hat{\Phi}$ included in $O(\cdot)$ in (3.6) is continuous and twice continuously differentiable except at the origin and its derivatives are integrable in a neighbourhood of the origin (they are therefore integrable over the whole real line, because of our assumptions about the asymptotic behaviour of $\hat{\phi}$ at infinity). It therefore makes a contribution of order $|x|^{-2}$ to $|\psi(x)|$ for large x. Thus, on account of the smoothness of $\tilde{\psi}$ away from the origin and because its derivatives are integrable over $\mathbb{R}\backslash(-\delta,\delta)$ for any positive δ, and in view of (3.6), the main contribution to the term on the left-hand side of (3.10) is contained in some constant multiple of

$$\left|\int_{-\infty}^{\infty} \sin(xt)\,\bar{\rho}(t)\left(\frac{1}{\log|t|}\right)' dt\right|, \tag{3.11}$$

where $\bar{\rho} \in \mathscr{C}^\infty(\mathbb{R})$ is a nonnegative function with support in $(-1/4, 1/4)$

and $\bar{\rho}\,|_{[-1/8,\,1/8]}=1$. Further, it will follow that (3.11) is of magnitude $O((\log|x|)^{-2})$ if we can show that

$$\int_0^{\pi} \sin(xt)\left(\frac{1}{\log(t/8\pi)}\right)' dt = -\int_0^{\pi} \sin(xt)\,\frac{dt}{t(\log(t/8\pi))^2} \tag{3.12}$$

is, because

$$\left|\int_{\mathbb{R}\setminus(-1/8,\,1/8)} \sin(xt)\,\bar{\rho}(t)\left(\frac{1}{\log|t|}\right)' dt\right| = O(|x|^{-1}), \qquad |x|>1,$$

as can be verified by integration by parts.

The non-oscillatory term in the integrand in (3.12) is strictly monotonically increasing since

$$-\left(\frac{1}{t(\log(t/8\pi))^2}\right)' = \frac{2+\log(t/8\pi)}{t^2(\log(t/8\pi))^3} > 0, \qquad 0 \leqslant t \leqslant \pi,$$

where we have used the fact that $\log\frac{1}{8} = -2.07944\ldots$. Therefore

$$\begin{aligned} 0 &< \int_0^{\pi} \sin(xt)\,\frac{dt}{t(\log(t/8\pi))^2} < \int_0^{\pi/x} \sin(xt)\,\frac{dt}{t(\log(t/8\pi))^2} \\ &\leqslant x \cdot \int_0^{\pi/x} \frac{dt}{(\log(t/8\pi))^2} = \frac{\pi}{(\log\xi)^2}, \end{aligned} \tag{3.13}$$

where $\xi \in (8x, \infty)$, so that (3.13) is indeed $O((\log|x|)^{-2})$ for large x, as required. Hence the theorem is true. ∎

A suitable convergence result to accompany Theorem 4 is the following:

THEOREM 5. *Let* ψ *be a continuous univariate function which satisfies* (3.8) *and*

$$\sum_{j=-\infty}^{\infty} \psi(x-j) = 1 \qquad \textit{for all} \quad x \in \mathbb{R}.$$

Then, for any bounded and uniformly continuous function f,

$$\|Q_h f - f\|_\infty = o(1), \qquad h \to 0.$$

Proof. This proof is a trivially modified version of the proof of Theorem 8.1 in Powell [15]. Therefore we will be brief. We have already noted that (3.8) implies that

$$\sup_{x\in\mathbb{R}} \sum_{j=-\infty}^{\infty} |\psi(x-j)| \tag{3.14}$$

is finite; call this number M. Also define

$$\sigma^*(t)=\sup_{x\in\mathbb{R}}\sum_{\{j:\,|x-j|>t\}}|\psi(x-j)|,\qquad t>0.$$

Observe that $\sigma^*(t)=o(1)$ as $t\to\infty$. This is a consequence of the uniform convergence of the sum in (3.14) which follows from our remarks in the previous proof. Now, let $\varepsilon>0$ be given. By the uniform continuity of f, there exist a positive δ such that $|f(x)-f(y)|\leqslant\varepsilon$ for all $|x-y|\leqslant\delta$. Therefore, given any real number x, we can estimate

$$\begin{aligned}|Q_hf(x)-f(x)|&=\left|\sum_{j=-\infty}^{\infty}\{f(jh)-f(x)\}\,\psi(h^{-1}x-j)\right|\\&\leqslant\sum_{\{j:\,|x-jh|\leqslant\delta\}}|f(jh)-f(x)|\cdot|\psi(h^{-1}x-j)|\\&\quad+\sum_{\{j:\,|x-jh|>\delta\}}|f(jh)-f(x)|\cdot|\psi(h^{-1}x-j)|\\&\leqslant M\varepsilon+2\,\|f\|_\infty\,\sigma^*(h^{-1}\delta)\leqslant(M+2\,\|f\|_\infty)\,\varepsilon\end{aligned}$$

for small enough h. Because ε was chosen arbitrarily, the theorem is therefore true. ∎

We conclude this section by making two observations about interpolation with radial functions in one dimension. Firstly we note that the same techniques as those in the proof of Theorem 4 can be applied to show the following theorem which complements the work on cardinal interpolation in [4], because logarithmic singularities in $\hat{\phi}$ akin to the one in (3.1) were excluded in that work.

THEOREM 6. *Let ϕ be a radial function of the type required in Theorem 4 which also satisfies $\hat{\phi}>0$. Then the inverse Fourier transform χ of the absolutely integrable function*

$$\frac{\hat{\phi}(|t|)}{\sum_{l=-\infty}^{\infty}\hat{\phi}(|t+2\pi l|)},\qquad t\in\mathbb{R},\tag{3.15}$$

satisfies $\chi(j)=\delta_{0j}$ for all integers j, it satisfies

$$|\chi(x)|=O\left(\frac{1}{|x|\,(\log|x|)^2}\right),\qquad |x|>2,\tag{3.16}$$

it satisfies

$$\sum_{j=-\infty}^{\infty} \chi(x-j)=1 \qquad \textit{for all} \quad x \in \mathbb{R}, \tag{3.17}$$

and it can be represented as an absolutely convergent sum

$$\chi(x)=\sum_{j=-\infty}^{\infty} d_j \phi(|x-j|), \qquad x \in \mathbb{R}, \tag{3.18}$$

where the $\{d_j\}_{j=-\infty}^{\infty}$ *are the Fourier coefficients of*

$$\left\{\sum_{l=-\infty}^{\infty} \hat{\phi}(|t+2\pi l|)\right\}^{-1}, \qquad t \in \mathbb{T}. \tag{3.19}$$

Proof. The expression (3.15) is absolutely integrable by the requirements we have made about $\hat{\phi}$. Therefore χ is well-defined and continuous and satisfies $\chi(j)=\delta_{0j}$ for all j, see Proposition 3 in [4]. We will now show that the estimate (3.16) is indeed a consequence of the same reasoning which we have applied in the penultimate proof: Call the function defined in (3.19) g. Then we have for small $|t|$

$$\begin{aligned} g(t)\,\hat{\phi}(|t|) &= \frac{1}{1+\hat{\phi}(|t|)^{-1}\sum_{l=-\infty}^{\prime\infty} \hat{\phi}(|t+2\pi l|)} \\ &= 1+\sum_{k=1}^{\infty}(-1)^k\left\{\frac{\sum_{l=-\infty}^{\prime\infty} \hat{\phi}(|t+2\pi l|)}{\hat{\phi}(|t|)}\right\}^k \qquad (3.20) \\ &= 1+\sum_{k=1}^{\infty}(-1)^k\left\{\frac{\sum_{l=-\infty}^{\prime\infty} \hat{\phi}(|2\pi l|)}{\hat{\phi}(|t|)}\right\}^k + O\left(-\frac{|t|}{\log|t|}\right), \qquad (3.21) \end{aligned}$$

where the dashed sum indicates that we are omitting the $l=0$ term. The conclusion that (3.20) and (3.21) are the same depends on the continuous differentiability of $\sum_{l=-\infty}^{\prime\infty} \hat{\phi}(|\cdot+2\pi l|)$ near the origin and on the expression (3.1). The part of (3.20) included in the "O-term" in (3.21) makes a contribution of order $|x|^{-2}$ to $|\chi(x)|$ when $|x|$ is large for the same reasons as those which are explained in the paragaph that follows the displayed equation (3.10). The constant term in (3.21) contributes an exponentially decaying term to χ. Denoting $\sum' \hat{\phi}(|2\pi l|)$ by H, we therefore only consider the second term in (3.21) which can be rewritten as

$$\sum_{k=1}^{\infty}\left\{\frac{H}{\log|t|}\,\frac{1}{1-a(\log|t|)^{-1}}\right\}^k + O\left(\frac{|t|}{(\log|t|)^2}\right), \tag{3.22}$$

by applying (3.1) once more. Again, we may neglect the second term in (3.22) and consider the first one which can be reexpressed as

$$\sum_{k=1}^{\infty} H^k \sum_{m=0}^{\infty} \binom{-k}{m} \frac{(-a)^m}{(\log|t|)^{k+m}}. \tag{3.23}$$

According to the analysis of the penultimate proof, each $(\log|t|)^{-k-m}$ term contributes a fixed (i.e., independent of m and k) multiple of $(k+m)|x|^{-1}(\log|x|)^{-k-m-1}$ to $|\chi(x)|$ when x is large in modulus. Hence the total contribution that comes from (3.23) is a constant multiple of

$$\frac{1}{|x|\log|x|}\sum_{k=1}^{\infty} H^k \sum_{m=0}^{\infty} \binom{-k}{m} (k+m)\frac{(-a)^m}{(\log|x|)^{k+m}}$$

which is $O(|x|^{-1}(\log|x|)^{-2})$, as required. This settles (3.16). It has been shown in the penultimate proof that (3.16) secures absolute convergence of the sum on the left-hand side of (3.17). The identity (3.17) follows from Lemma 1 and the remark pertaining to the applicability of this lemma in the paragraph containing the displayed equation (3.9). Because g is square-integrable over $\mathbb{T}$, its Fourier coefficients are square-summable, whence the series on the right-hand side of (3.18) converges absolutely by the Cauchy–Schwarz inequality. However, we can actually identify the decay rate of the Fourier coefficients of g, because the analysis above can be modified to show also that the Fourier coefficients of g satisfy

$$|d_j| = O\left(\frac{1}{|j|(\log|j|)^2}\right), \qquad |j|>2.$$

This estimate has been shown before by Powell (1988, private communication). It once more implies the absolute convergence of the series on the right-hand side of (3.18). The identity (3.18) then follows from standard arguments; see [4]. The theorem is proved. ∎

Now, secondly, we observe that the existence of bounded cardinal functions for radial functions of the kind addressed above can also be established for data on $\mathbb{Z}_+ = \{0, 1, 2, 3, ...\}$ (this is a problem Schoenberg [18] already has studied in the context of univariate cardinal B-splines):

THEOREM 7. *Let ϕ be a radial function of the type required in Theorem 6. Then there are coefficients $\{d_{jk}\}_{k=0}^{\infty}$ for $j = 0, 1, 2, 3, \ldots$, such that each*

$$\chi_j(x) := \sum_{k=0}^{\infty} d_{jk}\phi(|x-k|), \qquad x\in\mathbb{R}, \qquad j=0,1,2,3,\ldots, \tag{3.24}$$

is an absolutely convergent series and satisfies

$$\chi_j(k) = \delta_{jk} \qquad \textit{for all} \quad j = 0, 1, 2, 3, \ldots \textit{ and all } k = 0, 1, 2, 3, \ldots. \tag{3.25}$$

Moreover, the χ_j are uniformly bounded (independently of j) and therefore interpolation to $f\colon \mathbb{R} \mapsto \mathbb{R}$ on $\mathbb{Z}_+$

$$\tilde{I}f(x) = \sum_{j=0}^{\infty} f(j)\,\chi_j(x), \qquad x \in \mathbb{R},$$

is well-defined as soon as the $\{f(j)\}_{j=0}^{\infty}$ are absolutely summable.

Proof. The problem of finding coefficients $\{d_{jk}\}_{k=0}^{\infty}$ for the cardinal functions $\{\chi_j\}_{j=0}^{\infty}$ corresponds to the problem of inverting the singly-infinite Toeplitz matrix $\{\phi(|j-k|)\}_{j,k=0}^{\infty}$. We therefore appeal to a theory on the inversion of such matrices developed by Calderón *et al.* [6], but we have to extend their approach which does not directly apply to the present setting. The crux of their theory is a factorization of the reciprocal of the *symbol* associated with the Toeplitz matrix. This symbol is in the present context

$$\sigma(t) := \sum_{k=-\infty}^{\infty} \phi(|k|)\exp(ikt), \qquad t \in \mathbb{T}. \tag{3.26}$$

The series in (3.26) is, due to our requirements on ϕ, a square-summable series. It can be viewed as the Fourier expansion of the absolutely integrable 2π-periodic function

$$g(t) = \sum_{l=-\infty}^{\infty} \hat{\phi}(|t+2\pi l|), \qquad t \in \mathbb{T},$$

because the dominated convergence theorem and the Fourier inversion formula allow us to argue

$$\begin{aligned} &\frac{1}{2\pi}\int_{\mathbb{T}} \sum_{l=-\infty}^{\infty} \hat{\phi}(|t+2\pi l|)\exp(ikt)\,dt \\ &\quad = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\phi}(|t|)\exp(ikt)\,dt = \phi(|k|), \qquad k \in \mathbb{Z}. \end{aligned}$$

It follows from Körner [12, Theorem 15.3(i), p. 58], that σ and g are the same on $\mathbb{T}\setminus\{0\}$. Therefore σ has a logarithmic singularity at the origin (but is continuous elsewhere) which explains why the theory of Calderón *et al.* fails to apply directly in our context: their work is restricted to Toeplitz matrices with bounded symbols. The fact that σ and g are the same except at 0 has two further consequences which are crucial for factoring $1/\sigma$. The first consequence is that σ is essentially (i.e., almost everywhere) bounded away from zero. The second consequence is that $\log \sigma$ is absolutely integrable and square-integrable over $\mathbb{T}$. It is absolutely integrable and square-integrable because $\log \sigma$ is continuous except at zero and because (3.1) implies that $\log \sigma(t)$ behaves like $\log \log t^{-1}$ near zero, which is square-integrable in a neighbourhood of the origin:

$$\begin{aligned}\int_0^{1/3} (\log \log t^{-1})^2 \, dt &= [t(\log \log t^{-1})^2]_{0^+}^{1/3} + 2 \int_0^{1/3} \frac{\log \log t^{-1}}{\log t^{-1}} \, dt \\ &= \frac{1}{3} (\log \log 3)^2 + 2 \int_{\log 3}^{\infty} \frac{\log s}{s} \exp(-s) \, ds \\ &< \infty.\end{aligned}$$

In order to factor $1/\sigma$ we first decompose $\log \sigma$. To do this, we observe that $\log \sigma \in L^2(\mathbb{T})$ means that it has a square-summable Fourier expansion

$$\sum_{k=-\infty}^{\infty} a_k \exp(ikt), \qquad t \in \mathbb{T}. \tag{3.27}$$

The series (3.27) agrees with $\log \sigma$ everywhere except at zero, because $\log \sigma \in \mathscr{C}^2(\mathbb{T} \setminus \{0\})$ and because the *Dini Test* [20, p. 52] implies that an absolutely integrable periodic function has a Fourier expansion which agrees with the function wherever the function is differentiable. Therefore, (3.27) is essentially bounded from below. We now let

$$f_+(t) = \sum_{k=0}^{\infty} a_k \exp(ikt), \qquad t \in \mathbb{T},$$

and

$$f_-(t) = \sum_{k=-\infty}^{-1} a_k \exp(ikt), \qquad t \in \mathbb{T}.$$

The coefficients $\{d_{jk}\}_{k=0}^{\infty}$, $j = 0, 1, 2, 3, \ldots$, we seek will arise out of the factorization $1/\sigma = 1/\sigma_+ \times 1/\sigma_-$, where $\sigma_+ = \exp(f_+)$ and $\sigma_- = \exp(f_-)$ and both of these functions are essentially bounded away from zero. Note that $(\sigma_+)^{\pm 1}$ and $(\sigma_-)^{\pm 1}$ only have Fourier coefficients with nonnegative or negative indices, respectively, alike f_+ and f_-. This is an easy consequence of the orthogonality of the exponential functions $\exp(ijt)$ if we expand $\exp(\pm f_\pm)$ into a power series.

Following Caldéron *et al.*, we find it convenient to introduce the following operators between sequence spaces in order to identify the required coefficients $\{d_{jk}\}_{k=0}^{\infty}$. First of all we introduce the canonical projections

$$P: \begin{cases} l^p(\mathbb{Z}) & \mapsto l^p(\mathbb{Z}_+) \\ \{x_j\}_{j=-\infty}^{\infty} & \mapsto \{x_j\}_{j=0}^{\infty} \end{cases},$$

where p can be any number in the range $[1, \infty]$. Here, $l^p(Z)$ denotes the set of sequences indexed on the set Z (which can be either the set of integers, nonnegative integers, or negative integers), whose p-norm is finite. Secondly, for every essentially bounded $f\colon \mathbb{T} \mapsto \mathbb{C}$ we define M_f to be the operator

$$M_f: \begin{cases} l^2(\mathbb{Z}) & \mapsto & l^2(\mathbb{Z}) \\ \{x_j\}_{j=-\infty}^{\infty} & \mapsto & \{\sum_{k=-\infty}^{\infty} f_{j-k} x_k\}_{j=-\infty}^{\infty} \end{cases},$$

where $\{f_k\}_{k=-\infty}^{\infty}$ are the Fourier coefficients of f. We have $M_f\, l^2(\mathbb{Z}) \subseteq l^2(\mathbb{Z})$ for every essentially bounded f because $M_f \{x_j\}_{j=-\infty}^{\infty}$ is nothing else than the sequence of Fourier coefficients of f times the Fourier series of $\{x_j\}_{j=-\infty}^{\infty}$. This product is square integrable due to f's essential boundedness.

For every $f \in L^2(\mathbb{T})$ we denote also by M_f the operator

$$M_f: \begin{cases} l^2(\mathbb{Z}) & \mapsto & l^\infty(\mathbb{Z}) \\ \{x_j\}_{j=-\infty}^{\infty} & \mapsto & \{\sum_{k=-\infty}^{\infty} f_{j-k} x_k\}_{j=-\infty}^{\infty} \end{cases}.$$

The inclusion $M_f\, l^2(\mathbb{Z}) \subseteq l^\infty(\mathbb{Z})$ follows from the Cauchy–Schwarz inequality because the Fourier coefficients of f are square-summable. M_f can of course also operate on $l^2(\mathbb{Z}_+)$ and on $l^2(\mathbb{Z}_-)$ ($\mathbb{Z}_- := \{\cdots, -3, -2, -1\}$) sequences, where we view both $l^2(\mathbb{Z}_+)$ and $l^2(\mathbb{Z}_-)$ as subspaces of $l^2(\mathbb{Z})$ with the canonical inclusions

$$\begin{array}{ccc} l^2(\mathbb{Z}_+) & \hookrightarrow & l^2(\mathbb{Z}) \\ \{x_j\}_{j=0}^{\infty} & \mapsto & \{\ldots, 0, 0, x_0, x_1, \ldots\} \end{array}$$

and

$$\begin{array}{ccc} l^2(\mathbb{Z}_-) & \hookrightarrow & l^2(\mathbb{Z}) \\ \{x_j\}_{j=-\infty}^{-1} & \mapsto & \{\ldots, x_{-2}, x_{-1}, 0, 0, \ldots\} \end{array}.$$

We claim that the singly-infinite vector $\{d_{jk}\}_{k=0}^{\infty} := PM_{1/\sigma_*} PM_{1/\sigma}\, \delta_j$, where $\delta_j = \{\delta_{jk}\}_{k=0}^{\infty}$ and $j = 0, 1, 2, 3, \ldots$, renders χ_j an absolutely convergent series which gives (3.25). The absolute convergence of the right-hand side of (3.24) is clear by the Cauchy–Schwarz inequality because the vectors we have just defined are in $l^2(\mathbb{Z}_+)$ by virtue of the essential boundedness of both $1/\sigma_+$ and $1/\sigma_-$. In order to establish (3.25), we need to show that

$$\left\{\sum_{k=0}^{\infty} d_{jk}\phi(|l-k|)\right\}_{l=0}^{\infty}=\delta_j, \qquad j=0,1,2,3,..., \tag{3.28}$$

or, in the operator notation introduced above,

$$PM_g PM_{1/\sigma_+} PM_{1/\sigma_-}\delta_j=\delta_j.$$

Indeed, we have

$$\begin{aligned} PM_g PM_{1/\sigma_+} PM_{1/\sigma_-}\delta_j &= PM_g M_{1/\sigma_+} PM_{1/\sigma_-}\delta_j && (3.29)\\ &= PM_{\sigma_-} PM_{1/\sigma_-}\delta_j \\ &= PM_{\sigma_-} M_{1/\sigma_-}\delta_j - PM_{\sigma_-}(1-P) M_{1/\sigma_-}\delta_j && (3.30)\\ &= \delta_j - PM_{\sigma_-}(1-P) M_{1/\sigma_-}\delta_j && (3.31)\\ &= \delta_j, \end{aligned}$$

where (3.29) is true because M_{1/σ_+} leaves $l^2(\mathbb{Z}_+)$ invariant, i.e., $PM_{1/\sigma_+}P=M_{1/\sigma_+}P$, where in (3.30) and (3.31) 1 denotes the identity operator on $l^2(\mathbb{Z})$, and where (3.31) is the same as δ_j because M_{σ_-} leaves $l^2(\mathbb{Z}_-)$ invariant, i.e., the second term in (3.31) vanishes. Moreover, the identities $M_g M_{1/\sigma_+}=M_{\sigma_-}$ and $M_{\sigma_-} M_{1/\sigma_-}=1$ which we have used above can be verified as follows. Take, e.g., the first identity. There is a sequence $\{g_k\}_{k=1}^{\infty}$ of continuous and positive functions on $\mathbb{T}$ that converges to g in the $L^2(\mathbb{T})$-norm and that is also such that $\{1/g_k\}_{k=1}^{\infty}$ converges uniformly to $1/g$. For instance, it is sufficient to choose

$$g_k(t):=\begin{cases} g(t) & \text{if } |t|>k^{-1}, \\ g(k^{-1}) & \text{otherwise,}\end{cases} \qquad t\in\mathbb{T}.$$

Now, we can factor the g_k in the same fashion as above and we obtain factors $\sigma_{+,k}$ and $\sigma_{-,k}$ which converge in the $L^2(\mathbb{T})$-norm to σ_+ and σ_-, respectively. By their construction, they are also such that $1/\sigma_{+,k}$ and $1/\sigma_{-,k}$ converge almost everywhere uniformly to $1/\sigma_+$ and $1/\sigma_-$, respectively. Due to the boundedness and positivity of g_k and $\sigma_{\pm,k}$, we have $M_{g_k}M_{1/\sigma_{+,k}}=M_{\sigma_{-,k}}$ for all k, because all operators in this identity map $l^2(\mathbb{Z})$ into $l^2(\mathbb{Z})$. Now, let $x\in l^2(\mathbb{Z})$ be arbitrary. Then $M_{g_k}x\xrightarrow{k\to\infty}M_g x$ componentwise, because the Fourier coefficients of the g_k tend to the ones of g and because the Fourier coefficients of g_k and of g are in $l^2(\mathbb{Z})$. The same reasoning explains why for any $x\in l^2(\mathbb{Z})$ $M_{\sigma_{-,k}}x\xrightarrow{k\to\infty}M_{\sigma_-}x$. Further, we could show $M_{1/\sigma_{+,k}}x\xrightarrow{k\to\infty}M_{1/\sigma_+}x$ in the same way, but this follows even more directly from the essential uniform convergence of $1/\sigma_{+,k}$ to $1/\sigma_+$. Hence the first identity is settled and similar arguments show that the second one holds. Therefore (3.28) is true.

Finally, we note that

$$\sum_{k=0}^{\infty} |d_{jk}|^2 \leqslant \int_{\mathbb{T}} \sigma_+(t)^{-2} \sigma_-(t)^{-2}\, dt < \infty,$$

i.e., the $l_2(\mathbb{Z}_+)$-norm of the $\{d_{jk}\}_{k=0}^{\infty}$ is bounded independently of $j=0, 1, 2, 3, \ldots$. Therefore, by the Cauchy–Schwarz inequality, the χ_j are uniformly bounded. This shows that the theorem is true. ∎

We finally remark that it can be shown (a paper by the author is in preparation) that square-summability of the $\{f(j)\}_{j=0}^{\infty}$ suffices for the absolute summability of the series $\tilde{I}f(x)$.

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